Factoring multivariate integral polynomials

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Abstract.

We present an algorithm to factor polynomials in several variables with integral coefficients that is polynomial-time in the degrees of the polynomial to be factored. Our algorithm generalizes the algorithm presented in [7] to factor integral polynomials in one variable.

1. Introduction.

The problem of factoring polynomials with integral coefficients remained open for a long time, i.e. no polynomial-time factoring algorithm was known. The best known algorithms took exponential-time in the worst case; these algorithms had to consider a possibly exponential number of combinations of p-adic factors before the true factors could be found or irreducibility could be decided. In [1] it was proven that the problem of factorization in $\mathbb{Z}[X]$ belongs to NP $\cap$ co-NP, which made its membership of P quite likely [2]. That this was indeed the case, was proven in [7] where a polynomial-time algorithm for factoring in $\mathbb{Z}[X]$ was given. This algorithm is based on the following three observations:

(1.1) The multiples of degree $< m$ of a p-adic factor together form a lattice in $\mathbb{Z}^m$;

(1.2) If this p-adic factor is computed up to a high enough precision, then the factor we are looking for is the shortest vector in this lattice;

(1.3) An approximation of the shortest vector in such a lattice can be found in polynomial-time by means of the so-called *basis reduction algorithm*.

In this paper we show that (1.1) and (1.2) can be generalized to polynomials in $\mathbb{Z}[X_1, X_2, \ldots, X_t]$ in an elementary way, for any $t \geq 2$. Combined with the same basis reduction algorithm as in (1.3), this leads to a polynomial-time algorithm for factoring in $\mathbb{Z}[X_1, X_2, \ldots, X_t]$. In [8, 9, 10] we show that the above three points can be applied to various other kinds of polynomial factoring problems as well (like multi-

variate polynomials over finite fields or over algebraic number fields). Another approach to multivariate integral polynomial factorization is given in [5]. There the multivariate case is first reduced in polynomial-time to the bivariate case, next bivariate is reduced to univariate.

For practical purposes we do not recommend any of these polynomial-time algorithms; their running time will be dominated by the rather slow basis reduction algorithm. For polynomials in $\mathbb{Z}[X_1, X_2, \ldots, X_t]$ the algorithm from [12] for instance is very useful, although it is exponential-time in the worst case.

We restrict ourselves in this paper to integral polynomials in two variables; the multivariate case follows immediately from this. In Section 2 we present an important result from [7: Section 1] concerning the basis reduction algorithm mentioned in (1.3). The generalizations of (1.1) and (1.2) to polynomials in $\mathbb{Z}[X,Y]$ are described in Section 3, and in Section 4 we give an outline of the factoring algorithm, and we analyze its running time.

2. The basis reduction algorithm.

The basis reduction algorithm from [7: Section 1] makes it possible to determine in polynomial-time a reasonable approximation of the shortest vector in a lattice. We will not give a description of the algorithm here. It will suffice to summarize those results from [7: Section 1] that we will need here.

Let $b_1, b_2, \ldots, b_n \in \mathbb{Z}^n$ be linearly independent. For our purposes we may assume that the $n \times n$ matrix having $b_1, b_2, \ldots, b_n$ as columns is upper-triangular. The i-dimensional lattice $L_i \subset \mathbb{Z}^i$ with basis $b_1, b_2, \ldots, b_i$ is defined as $L_i = \Sigma_{j=1}^{i} \mathbb{Z} b_j = \{\Sigma_{j=1}^{i} r_j b_j : r_j \in \mathbb{Z}\}$. We put $L = L_n$.

(2.1) Proposition. (cf. [7: (1.11), (1.26), (1.37)]) Let $B \in \mathbb{Z}_{\geq 2}$ be such that $|b_j|^2 \leq B$ for $1 \leq j \leq n$, where $|\ |$ denotes the ordinary Euclidean length. The basis reduction algorithm as described in [7: (1.15)] determines a vector $\tilde{b} \in L$ such that $\tilde{b}$ belongs to a basis for L, and such that $|\tilde{b}|^2 \leq 2^{n-1}|x|^2$ for every $x \in L$, $x \neq 0$; the algorithm takes $O(n^4 \log B)$ elementary operations on integers having binary length $O(n \log B)$. Furthermore, during the first $O(i^4 \log B)$ operations (on integers having binary length $O(i \log B)$), vectors $\tilde{b}_i \in L_i$, belonging to a basis for L_i, are deter-

mined such that $|\tilde{b}_i|^2 \le 2^{i-1}|x_i|^2$ for every $x_i \in L_i$, $x_i \neq 0$, for $1 \le i \le n$. □

So, we can find a reasonable approximation of the shortest vector in L in polynomial-time. But also we find, during this computation, approximations of the shortest vectors of the lattices L_i without any time loss.

3. Factors and lattices.

We describe how to generalize (1.1) and (1.2) to polynomials in $\mathbb{Z}[X,Y]$. Let $f \in \mathbb{Z}[X,Y]$ be the polynomial to be factored; we may assume that f has no multiple factors, i.e. f is *square-free*. Furthermore we assume that f is *primitive* with respect to X, i.e. the greatest common divisor of the coefficients in $\mathbb{Z}[Y]$ of f equals one. We denote by $\delta_X f$ and $\delta_Y f$ the degrees of f in X and Y respectively, and by $\ell c(f)$ the *leading coefficient* of f with respect to X. We put $n_X = \delta_X f$ and $n_Y = \delta_Y f$.

Suppose that we are given a prime number p, an integer s and a positive integer k. By (s_1) we denote the ideal generated by p and $(Y-s)$, and by (s_k) we denote the ideal generated by p^k and $(Y-s)^{n_Y+1}$. In Section 4 we will see how to find a polynomial $h \in \mathbb{Z}[X,Y]$ such that:

(3.1) $\ell c(h) = 1$,

(3.2) $(h \bmod (s_k))$ divides $(f \bmod (s_k))$ in $\mathbb{Z}[X,Y]/(s_k)$,

(3.3) $(h \bmod (s_1)) \in (\mathbb{Z}/p\mathbb{Z})[X]$ is irreducible in $(\mathbb{Z}/p\mathbb{Z})[X]$,

(3.4) $(h \bmod (s_1))^2$ does not divide $(f \bmod (s_1))$ in $(\mathbb{Z}/p\mathbb{Z})[X]$.

We put $\ell = \delta_X h$; so $0 < \ell \le n_X$.

Let $h_0 \in \mathbb{Z}[X,Y]$ be the irreducible factor of f for which $(h \bmod (s_1))$ divides $(h_0 \bmod (s_1))$ in $(\mathbb{Z}/p\mathbb{Z})[X]$ (or equivalently $(h \bmod (s_k))$ divides $(h_0 \bmod (s_k))$ in $\mathbb{Z}[X,Y]/(s_k)$, cf. [7: (2.5)]); notice that h_0 is unique up to sign.

(3.5) Let m_X and m_Y be two integers with $\ell \le m_X < n_X$ and $0 \le m_Y \le \delta_Y \ell c(f)$. We define L as the collection of polynomials $g \in \mathbb{Z}[X,Y]$ such that

(i) $\delta_X g \le m_X$,

(ii) $\delta_Y g \le n_Y$,

(iii) $\delta_Y \ell c(g) \le m_Y$,

(iv) $(h \bmod (s_k))$ divides $(g \bmod (s_k))$ in $\mathbb{Z}[X,Y]/(s_k)$.

Putting $M = m_X(n_Y+1)+m_Y+1$ it is not difficult to see that L is an M-dimensional lattice contained in $\mathbb{Z}^M$, where we identify polynomials in L and M-dimensional vectors in the usual way (i.e. $\sum_{i=0}^{m_X-1}\sum_{j=0}^{n_Y} a_{ij}X^iY^j + \sum_{j=0}^{m_Y} a_{m_Xj}X^{m_X}Y^j$ is identified with $(a_{00}, a_{01}, \ldots, a_{0n_Y}, a_{10}, \ldots, a_{m_X-1\,n_Y}, a_{m_X0}, \ldots, a_{m_Xm_Y})$). Because of (3.1) a basis for L is given by

$$\{p^{k_Y}j_X^i: \ 0 \le j \le n_Y, \ 0 \le i < \ell\} \ \cup$$

$$\{(hY^j \bmod (s_k))X^{i-\ell}: \ (0 \le j \le n_Y \text{ and } \ell \le i < m_X) \text{ or } (0 \le j \le m_Y \text{ and } i = m_X)\}.$$

This generalizes (1.1) (cf. [7: (2.6)]). We now come to (1.2). The *height* g_{max} of a polynomial g is defined as the maximal absolute value of any of its integral coefficients. We prove that, if k and s are suitably chosen, then a vector of small height in L must lead to a factorization of f.

(3.6) Proposition. Suppose that $g \in L$ satisfies

$$|s|^{n_Y+1} > (e^{n_X+n_Y} f_{max}\sqrt{(n_X+1)(n_Y+1)})^{m_X}(g_{max}\sqrt{(m_X+1)(n_Y+1)})^{n_X} \tag{3.7}$$

and

$$p^k > (e^{n_X+n_Y} f_{max}\sqrt{(n_X+1)(n_Y+1)})^{m_X}(g_{max}\sqrt{(m_X+1)(n_Y+1)})^{n_X}(1+(1+|s|)^{n_Y+1})^{n_Y(n_X+m_X-1)}. \tag{3.8}$$

Then h_0 divides g in $\mathbb{Z}[X,Y]$, and in particular $\gcd(f,g) \neq 1$.

Proof. Suppose that $\gcd(f,g) = 1$. This implies that the resultant $R \in \mathbb{Z}[Y]$ of f and g is unequal to zero. Using the result from [4] one proves that

$$|R| < (f_{max}\sqrt{(n_X+1)(n_Y+1)})^{m_X}(g_{max}\sqrt{(m_X+1)(n_Y+1)})^{n_X}, \tag{3.9}$$

where $|R|$ denotes the ordinary Euclidean length of the vector identified with R. Since $(h \bmod (s_k))$ divides both $(f \bmod (s_k))$ and $(g \bmod (s_k))$, the polynomials f and g have a non-trivial common divisor in $\mathbb{Z}[X,Y]/(s_k)$, so that R must be zero modulo the ideal generated by p^k and $(Y-s)^{n_Y+1}$. The polynomial $(Y-s)^{n_Y+1}$ cannot divide R, because this would imply, according to [11: Theorem 1], that $|s|^{n_Y+1} \le |R|$, which is, combined with (3.9), a contradiction with (3.7). Therefore $(R \bmod (Y-s)^{n_Y+1})$ has to be zero modulo p^k. Using induction on n_Y+1 it is easy to prove that

$$(R \bmod (Y-s)^{n_Y+1})_{max} \le R_{max}(1+(1+|s|)^{n_Y+1})^{n_Y(n_X+m_X-1)},$$

so that, with $R_{max} \le |R|$ and (3.8), it follows that $(R \bmod (Y-s)^{n_Y+1})$ cannot be zero

modulo p^k. We conclude that $\gcd(f,g) \neq 1$.

Suppose that h_0 does not divide g. So h_0 does not divide $r = \gcd(f,g)$, so $(h \bmod (s_k))$ divides $((f/r) \bmod (s_k))$. Because f/r divides f, we find from [3] that $(f/r)_{max} \leq e^{n_X+n_Y} f_{max}$. This implies that the above reasoning applies to f/r and the same polynomial g in L, so that $\gcd(f/r,g) \neq 1$. This is a contradiction with $r = \gcd(f,g)$, because f is square-free. □

(3.10) Proposition. Suppose that s and k are chosen in such a way that (3.7) and (3.8) are satisfied with g_{max} replaced by $2^{(M-1)/2}\sqrt{M}\, e^{n_X+n_Y} f_{max}$. Let $\tilde{b}$ be as in (2.1) the result of an application of the basis reduction algorithm to the M-dimensional lattice L as defined in (3.5). Then $h_0 \in L$ if and only if (3.7) and (3.8) are satisfied with g replaced by $\tilde{b}$.

Proof. To prove the "if"-part, assume that (3.7) and (3.8) hold with g_{max} replaced by $\tilde{b}_{max}$. According to (3.6) this implies that h_0 divides $\tilde{b}$, so that $h_0 \in L$.

To prove the "only if"-part, assume that $h_0 \in L$. Because h_0 divides f, we find from [3] that $(h_0)_{max} \leq e^{n_X+n_Y} f_{max}$. So there exists a non-zero vector in L with Euclidean length bounded by $\sqrt{M}\, e^{n_X+n_Y} f_{max}$. Application of (2.1) yields that $\tilde{b}_{max} \leq |b| \leq 2^{(M-1)/2}\sqrt{M}\, e^{n_X+n_Y} f_{max}$. Combined with the above choices of s and k, this implies that (3.7) and (3.8) hold with g replaced by $\tilde{b}$. □

4. Description of the algorithm.

In this section we present the polynomial-time algorithm to factor f. First we give an algorithm to determine the factor h_0, given p, s and h. After that, we will see how p and s have to be chosen.

(4.1) Let p, s and h be as in Section 3, such that (3.1), (3.3), (3.4) and (3.2) with k replaced by 1 are satisfied. Assume that s satisfies the condition in (3.10) with m_X and m_Y replaced by n_X-1 and $\delta_Y \ell c(f)$ respectively:

$$(4.2) \qquad |s|^{n_Y+1} > (e^{n_X+n_Y} f_{max} \sqrt{(n_X+1)(n_Y+1)})^{n_X-1} (2^{(M-1)/2}\sqrt{M}\, e^{n_X+n_Y} f_{max} \sqrt{n_X(n_Y+1)})^{n_X}$$

where $M = (n_X-1)(n_Y+1)+\delta_Y \ell c(f)+1$. We describe an algorithm that determines h_0, the irreducible factor of f such that $(h \bmod (s_1))$ divides $(h_0 \bmod (s_1))$ in $(\mathbb{Z}/p\mathbb{Z})[X]$.

We may assume that $\ell = \delta_X h < n_X$. Take k minimal such that the condition from (3.10) is satisfied with m_X and m_Y replaced by n_X-1 and $\delta_Y \ell c(f)$ respectively:

$$(4.3) \qquad p^k > (e^{n_X+n_Y} f_{max} \sqrt{(n_X+1)(n_Y+1)})^{n_X-1} (2^{(M-1)/2} \sqrt{M}\, e^{n_X+n_Y} f_{max} \sqrt{n_X(n_Y+1)})^{n_X} \cdot (1+(1+|s|)^{n_Y+1})^{2n_Y(n_X-1)}.$$

Next modify h in such a way that (3.2) also holds for this value of k; because of (3.4) this can be done by means of Hensel's lemma [13].

Apply Proposition (2.1) to the M-dimensional lattice L as defined in (3.5) for each of the values of $M = \ell(n_Y+1)+1, \ell(n_Y+1)+2, \ldots, \ell(n_Y+1)+\delta_Y \ell c(f)+1, (\ell+1)(n_Y+1)+1, \ldots, (n_X-1)(n_Y+1)+\delta_Y \ell c(f)+1$ in succession (so, for $m_X = \ell, \ell+1, \ldots, n_X-1$ in succession and for every value of m_X the values $m_Y = 0, 1, \ldots, \delta_Y \ell c(f)$ in succession). But stop as soon as a vector $\tilde{b}$ is found satisfying (3.7) and (3.8) with g replaced by $\tilde{b}$.

If such a vector $\tilde{b}$ is found for a certain value of M ($m_X = m_{X0}$ and $m_Y = m_{Y0}$), then we know from (3.10) that $h_0 \in L$. Since we try the values of M in succession this implies that $\delta_X h_0 = m_{X0}$ and $\delta_Y \ell c(h_0) = m_{Y0}$. By (3.6) h_0 divides $\tilde{b}$, so that $\delta_X \tilde{b} = m_{X0}$ and $\delta_Y \ell c(\tilde{b}) = m_{Y0}$. So $\tilde{b} = c h_0$ for some $c \in \mathbb{Z}$, but $h_0 \in L$ and $\tilde{b}$ belongs to a basis for L, so $\tilde{b} = \pm h_0$.

If no such vector $\tilde{b}$ was found, then (3.10) implies that $\delta_X h_0 > n_X-1$, so that $h_0 = f$, because f is primitive.

This finishes the description of Algorithm (4.1).

(4.4) Proposition. Denote by $m_{X0} = \delta_X h_0$ the degree in X of the irreducible factor h_0 of f that is found by Algorithm (4.1). Then the number of arithmetic operations needed by Algorithm (4.1) is $O(m_{X0}(n_X^5 n_Y^5 + n_X^4 n_Y^4 \log(f_{max}) + n_X^4 n_Y^6 \log(|s|) + n_X^3 n_Y^4 \log p))$ and the integers on which these operations have to be performed each have binary length $O(n_X^3 n_Y^2 + n_X^2 n_Y \log(f_{max}) + n_X^2 n_Y^3 \log(|s|) + n_X n_Y \log p)$.

Proof. Let M_1 be the largest value of M for which (2.1) is applied; so $M_1 = O(m_{X0} n_Y)$. It follows from (2.1) that the number of operations needed for the applications of the basis reduction algorithm for $\ell(n_Y+1)+1 \le M \le M_1$ is equal to the number of operations needed for $M = M_1$ only. Assuming that the coefficients of the initial basis for L are reduced modulo p^k, we find, using (4.3), that the following holds for the bound

B on the length of these vectors:

$$\log B = O(n_X^2 n_Y + n_X \log(f_{max}) + n_X n_Y^2 \log(|s|) + \log p).$$

With $M_1 = O(m_{X0} n_Y)$ and (2.1) this gives the estimates in (4.4).

The verification that the same estimates are valid for the application of Hensel's lemma is straightforward [13]. □

We now describe how s and p have to be chosen. First, s must be chosen such that $(f \bmod (Y-s)) = f(X,s)$ remains square-free, and such that (4.2) holds. The resultant R of f and its derivative f' with respect to X is a non-zero polynomial in $\mathbb{Z}[Y]$ of degree $\leq n_Y(2n_X-1)$. Therefore we can find in $O(n_X n_Y)$ trials the minimal integer s such that s is not a zero of R, and such that (4.2) holds. It is easily verified that $\log(|s|) = O(n_X^2 + n_X \log(f_{max}))$.

Next we choose p as the smallest prime number not dividing the resultant of $f(X,s)$ and $f'(X,s)$. Since $\log(f(X,s)_{max}) = O(n_X^2 n_Y + n_X n_Y \log(f_{max}))$, it follows as in the proof of [7: (3.6)] that $p = O(n_X^3 n_Y + n_X^2 n_Y \log(f_{max}))$.

The complete factorization of $(f \bmod (s_1))$ can be determined by means of Berlekamp's algorithm [6: section 4.6.2]; notice that (3.4) holds for every factor $(h \bmod (s_1))$ of $(f \bmod (s_1))$, because of the choice of p, and that this factorization can be found in polynomial-time, because of the bound on p. The algorithm to factor f completely now follows by repeated application of Algorithm (4.1). The above bounds on $\log(|s|)$ and p, combined with (4.4) and the fact that a factor g of f satisfies $\log(g_{max}) = O(n_X + n_Y + \log(f_{max}))$ (cf. [3]), yields the following theorem.

(4.5) Theorem. The number of arithmetic operations needed to factor f completely is $O(n_X^7 n_Y^6 + n_X^6 n_Y^6 \log(f_{max}))$, and the integers on which these operations have to be performed each have binary length $O(n_X^4 n_Y^3 + n_X^3 n_Y^3 \log(f_{max}))$. □

5. Conclusion.

We have shown that basically the same ideas that were used for the polynomial-time algorithm for factoring in $\mathbb{Z}[X]$ lead to a polynomial-time factoring algorithm in $\mathbb{Z}[X,Y]$ (Theorem (4.5)). Our method can be generalized to polynomials in $\mathbb{Z}[X_1, X_2,$

$\ldots, X_t]$. The evaluation $(Y = s)$ is then replaced by $(X_2 = s_2, X_3 = s_3, \ldots, X_t = s_t)$, where the integers s_i have to satisfy conditions similar to (4.2). It will not be surprising that in this case the estimates become rather complicated.

A somewhat simpler algorithm results if we use the algorithm from [7]; the details of this algorithm, which is similar to the one described in this paper, can be found in [10].

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